Statistical Theory of Extremes

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Part 4
Stochastic Sequences and Processes of Extremes

Chapter 15

Other Processes of Extremes: Basics and Statistics

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Abstract

Properties of the extreme-Markovian-stationary (EMS) sequences analogous are introduced in this chapter. Statistical decision for EMS sequences and processes and Extreme-Markovian-evolutionary (EME) sequences with some remarks on statistical decision are given. Sliding extreme (SE) sequences and statistical decision for SE sequences are also given with applications for earthquake analysis and modelling.

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15.1 Extreme-Markovian-stationary (EMS) sequences

Let us introduce the extreme-Markovian-stationary (EMS) sequences analogous to normal-Markovian-stationary ones and study some of their properties. In this section and in the following one will follow Tiago de Oliveira (1972).

Let $\{E_k, k = 1, 2, \dots\}$ be a sequence of independent reduced Gumbel random variables and consider the sequence

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$Z_1 = E_1$

$Z_k = \max(Z_{k-1} + a, E_k + b), k \geq 2.$

We see immediately that *all Z_k have distribution function $\Lambda(x)$ iff we have* $e^a + e^b = 1$. Putting $e^a = \theta (0 \leq \theta \leq 1)$ we get the EMS sequence defined as

$Z_1 = E_1$

$Z_k = \max(Z_{k-1} + \log\theta, E_k + \log(1-\theta)), k \geq 2 (0 \leq \theta \leq 1).$

The distribution function of the pair (Z_k, Z_{k+1}) $(k \geq 1)$ is Prob $\{Z_k \leq x, Z_{k+1} \leq y\} = \text{Prob}\{Z_k \leq x, Z_k \leq y - \log\theta, E_{k+1} \leq y - \log(1-\theta)\} = \Lambda(\min(x, y - \log\theta))\, \Lambda(y - \log(1-\theta)) = \Lambda(x, y|\theta) = \exp\{-(e^{-x} + e^{-y}) + \min(e^{-x}, \theta\, e^{-y})\}$ where the dependence function is $k(w|\theta) = 1 - \min(\theta, e^w)/(1 + e^w)$.

The transition distribution function $\text{Prob}\{Z_k \leq y | Z_{k-1} = x\}$ is given as

$P(y|x;\theta) = \text{Prob}\{\max(X + \log\theta, E_k + \log(1-\theta)\} \leq y) = 0$ if $y < x + \log\theta$ and $P(y|x;\theta) = \exp\{-(1-\theta)e^{-y}\}$ if $y \geq x + \log\theta$, that is $P(y|x;\theta) = H(y - x - \log\theta). \exp - \{(1-\theta)e^{-y}\}$, H being the Heaviside jump function. The transition density, using the Dirac δ pseudo-function, is

$[(1-\theta)e^{-y}\, H(y - x - \log\theta) + \delta(y - x - \log\theta)] \exp\{-(1-\theta)e^{-y}\}.$

The distribution function of $(Z_1, \dots, Z_n)$ is given by the iterative formula

$\Lambda_n(x_1, \dots, x_n|\theta) = \text{Prob}\{Z_1 \leq x_1, \dots, Z_n \leq x_n\} = \text{Prob}\{Z_1 \leq x_1, \dots, Z_{n-1} \leq x_{n-1}, Z_{n-1} \leq x_n - \log\theta, E_n \leq x_n - \log(1-\theta)\} = \Lambda_{n-1}(x_1, \dots, x_{n-2}, \min(x_{n-1}, x_n - \log\theta)|\theta)\, \Lambda(x_n - \log(1-\theta)).$

The distribution function of (Z_p, Z_q) $(p < q)$ is given by

$\Lambda_{p,q}(x, y|\theta) = \text{Prob}\{Z_p \leq x, Z_q \leq y\}$

$= \Lambda_{p,q-1}(x, y - \log\theta|\theta). \Lambda(y - \log(1-\theta))$ if $p < q - 1$,

$= \Lambda(\min(x, y - \log\theta))\ \Lambda(y - \log(1-\theta)) = \Lambda(x, y|\theta)$ if $p = q - 1$.

The iteration shows that the distribution function of (Z_p, Z_q) is $\Lambda(x, y|\theta^{q-p})$, so that the correlation coefficient between Z_p and Z_q is $\frac{6}{\pi^2} R(\theta^{q-p})$.

The expression of Z_q in $Z_p (q > p)$ is

$$Z_q = \max[Z_p + (q-p)\log\theta, \max_{p+1}^{q} (E_j + (q-j)\log\theta) + \log(1-\theta)].$$ Thus:

The EMS sequence Z_n is stationary with the distribution function given by the iteration

$$\Lambda(x|\theta) = \Lambda(x), \Lambda_n(x_1, \dots, x_n|\theta) =$$

$$\Lambda_{n-1}(x_1, \dots, x_{n-2}, min(x_{n-1}, x_n - log\,\theta)|\theta).\Lambda(x_n - log(1-\theta)).$$

Its mean value is γ, the variance $\pi^2/6$ and the correlation coefficient $6/\pi^2 . R(\theta)$; the correlation coefficient between Z_p and $Z_q (q > p)$ is $\frac{6}{\pi^2} R(\theta^{q-p})$.

The distribution function of $\max(Z_1, \dots, Z_n)$ being

$$\text{Prob}\{\max(Z_1, \dots, Z_n) \le x\} = \exp\{-(\theta + n(1-\theta))e^{-x}\},$$

we see that

$$\frac{\max(Z_1, \dots, Z_n)}{\log(\theta + n(1-\theta))} \xrightarrow{p} 1 \text{ as } n \to \infty.$$

Also $\frac{\max(Z_1, \dots, Z_n)}{\log n} \xrightarrow{p} 1$ and the mean value and variance of $\max(Z_1, \dots, Z_n)$ are $\gamma + \log(\theta + n(1-\theta))$ and $\pi^2/6$.

Let us now obtain some results which may be useful for the description and analysis of an EMS sequence.

Let us first show that:

The ergodic theorem in mean-square for the EMS sequence is valid.

As $M(\frac{1}{n}\sum_1^n Z_k) = \gamma$ we have only to show that $V(\frac{1}{n}\sum_1^n Z_k) \to 0$. But

$$V(\frac{1}{n}\sum_1^n Z_k) = \frac{\pi^2/6}{n^2}\sum_{i,j}^n \rho_{ij} = \frac{\pi^2/6}{n^2}(n + 2\sum_1^n (n-k)\,\rho_k)$$

where the correlation coefficient ρ_k is $\rho_{i,i+k} = 6/\pi^2 . C(Z_i, Z_{i+k})$ by the stationarity. As $\rho_k = \frac{6}{\pi^2} R(\theta^k)$ we have

$$\frac{1}{n}\sum_1^n (1 - \frac{k}{n})\, R(\theta^k) \to 0,$$

which shows the desired result.

The results that follow suggest the trend of the next section concerning statistical decision.

As the EMS sequence is stationary the correlogram technique can be applied; from $\theta^k \downarrow 0$ we can expect the empirical correlogram to tend quickly to zero.

By contrast, *the periodogram technique is not useful as could be expected*. Let T be the trial period. The quantities $A_n(T) = \frac{1}{n}\sum_1^n Z_k \sin\frac{2\pi k}{T}$ and $B_n(T) = \frac{1}{n}\sum_1^n Z_k \cos\frac{2\pi k}{T}$ converge in mean square to zero, as their mean values converge to zero and $V(A_n(T))$ and $V(B_n(T))$ converge to zero. A simple way is to show that

$$V(A_n(T)) + V(B_n(T)) = \frac{\pi^2/6}{n^2}\sum_{i,j}\rho_{ij}\,\cos\frac{2\pi(i-j)}{T} =$$

$$\frac{\pi^2/6}{n^2}(n+2)\sum_1^n \rho_k(n-k)\cos\frac{2\pi k}{T} \to 0,$$

which is evident because $\rho_k = 6/\pi^2\, R(\theta^k) \to 0$ (as follows from the ergodic theorem).

We can also consider the associated sequence of ups and downs. Denoting by K_n the number of times that $Z_{k-1} \le Z_k$ for $(Z_1, Z_2, \dots, Z_n)$ we have

$$K_n = \sum_2^n H(Z_k - Z_{k-1}).$$

As $Z_n = \max(Z_{k-1} + \log\theta, E_k + \log(1-\theta))$, we have $H(Z_k - Z_{k-1}) = H(E_k + \log(1-\theta) - Z_{k-1})$. Thus $M(\frac{K_n}{n-1}) = \frac{1-\theta}{2-\theta}$ and $V(\frac{K_n}{n-1}) = \frac{1}{(n-1)^2}[(n-1)\sigma^2 + 2\sum_1^n(1-\frac{k}{n-1})\rho'_k]$ with $\sigma^2 = V(H(Z_k - Z_{k-1})) = \frac{1-\theta}{(2-\theta)^2}$ and $\rho'_k = -\frac{(1-\theta)^2\theta^{k-1}}{1+(1-\theta)(1+\theta^{k-1})}$ as

$$C(H(Z_2 - Z_1), H(Z_{k+2} - Z_{k-1})) = \text{Prob}(Z_1, \le Z_2, Z_{k+1} \le Z_{k+2}) - (\frac{1-\theta}{2-\theta})^2$$

$$= -\frac{(1-\theta)^3\,\theta^{k-1}}{(2-\theta)^2(1+(1-\theta)(1+\theta^{k-1}))}.$$

As $\rho'_k \to 0$ when $k \to \infty$ *we see that* $\frac{K_n}{n-1} \xrightarrow{ms} \frac{1-\theta}{2-\theta}$.

We can generalize, by defining another type of stochastic sequences of extremes analogous to linear processes in gaussian processes theory.

Let $\{E_n, -\infty < n < +\infty\}$ be a sequence of independent reduced Gumbel random variables and define

$Z_k = \max_{0\le j<+\infty}(E_{k-j} + a_j)$.

Then $\operatorname{Prob}\{Z_k \le x\} = \operatorname{Prob}\{E_{k-j} \le x - a_j\} = \overset{\infty}{\underset{0}{\pi}} \Lambda(x - a_j) =$ $\exp\{-e^{-x} \sum_0^\infty e^a j\}$ has Gumbel reduced margins iff $\sum_0^\infty e^a j = 1$.

Taking then any set of probabilities $p_j \ge 0 (\sum_0^\infty p_j = 1)$, we must have

$Z_k = \max_{0\le j<\infty}(E_{k-j} + \log p_j)$,
the EMS sequence being obtained by taking $p_j = (1-\theta)\theta^j$.

For a sequence $\{E_n, 0 \le n < +\infty\}$ of independent reduced Gumbel random variables we can also define an extreme sequence by choosing $\beta_n(j) \ge 0$ such that $\sum_1^n \beta_n(j) = 1$ and putting

$Z_n = \max_{0\le k<n}(E_k + \log \beta_n(k))$.

15.2 Statistical decision for EMS sequences

Let us return to EMS sequences with reduced margins. From the relation $Z_k - Z_{k-1} \ge \log\theta$ $(k \ge 2)$ we get the maximum likelihood estimator

$\hat{\theta}_n = \exp(\min_{2\le k\le n}(Z_k - Z_{k-1}))$.

We can easily obtain the distribution of $\hat{\theta}_n$ and show that $\operatorname{Prob}\{\hat{\theta}_n = \theta\} \to 1$. Note that $\hat{\theta}_n \ge \theta$, but we can have $\hat{\theta}_n > 1$, thus suggesting truncation if this is the case.

Then $\operatorname{Prob}\{\hat{\theta}_n \le a\} = 0$ for $a < \theta$. Let us then obtain $\operatorname{Prob}\{\hat{\theta}_n \le a\}$ for $a \ge \theta$, if $\theta < 1$ by computing $Q_n(a) = \operatorname{Prob}\{\hat{\theta}_n > a\}$ for $a \ge \theta$.

As $Z_k - Z_{k-1} > \log a$ $(\log \ge \theta)$ we see that

$Q_n(a) = \operatorname{Prob}\{Z_k - Z_{k-1} > \log a\} =$

$\operatorname{Prob}\{E_2 > E_1 + \log\frac{a}{1-\theta}, E_3 > E_2 + \log a, \dots, E_n > E_{n-1} + \log a\} =$

$= \text{Prob}\{t_1 > a\, q_1(\theta) t_2, t_2 > a\, t_3, \dots, t_{n-1} > a\, t_n\}$,

with $t_i = e^{-E_i}$ standard exponential and $q_1(\theta) = (1-\theta)^{-1}$, which, iteratively, gives

$$Q_n(a) = \left(\prod_{2}^{n} q_k(a)\right)^{-1} \text{ where } q_k(a) = 1 + a\, q_{k-1}(a),$$

so that, as $q_l(a) = (1-\theta)^{-1}$, we have $q_k(a) = \frac{a^k}{\theta(1-\theta)} + \frac{1-a^{k-1}}{1-a}$, $q_k(\theta) = (1-\theta)^{-1}$,

$Q_n(\theta) = (1-\theta)^{n-1}$

and thus $\text{Prob}(\hat{\theta}_n = \theta) = 1 - (1-\theta)^{n-1} \to 1$ if $\theta > 0$ and $\text{Prob}(\hat{\theta}_n = 1) = 1$ if $\theta = 1$. This estimator has for $a = \theta$

$\frac{d\,\text{Prob}(\hat{\theta}_n \leq a)}{d\,a} \sim n(1-\theta)^{n-2}$ if $0 \leq \theta < 1$

$= 0 \quad \text{if} \quad \theta = 1;$

asymptotically $\hat{\theta}_n$ is better than the common maximum likelihood estimators whose order is, usually, $n^{1/2}$.

In the more general case, we have an EMS sequence $X_n = \lambda + \delta\, Z_n$ with general margins, where λ and $\delta (> 0)$ are the location and dispersion parameters. We will show, using the results given in the paper, how to obtain quick estimators for λ, δ and θ.

We have $\overline{X}_n = \lambda + \delta\, \overline{Z}_n \xrightarrow{P} \lambda + \gamma\, \delta$ by the ergodic results,

$$F_n = \frac{K_n}{n-1} \xrightarrow{P} \frac{1-\theta}{2-\theta}$$

by the ups and downs sequence,

$\Delta_n = \min(X_k - X_{k-1}) = \delta \min(Z_k - Z_{k-1}) \xrightarrow{P} \delta \log \theta$,

and also

$\frac{\max(X_1,\dots,X_n)}{\log n} = \frac{\lambda + \delta \max(Z_1,\dots,Z_n)}{\log n} \xrightarrow{P} \delta.$

As, for $0 \leq \theta \leq 1$, we have $0 \leq \frac{1-\theta}{2-\theta} \leq \frac{1}{2}$, decreasing with θ, we will take

$F_n^* = F_n$ if $0 \leq F_n \leq 1/2$

$F_n^* = 1/2$ if $1/2 \leq F_n (\leq 1)$,

and use

$F_n^* = \frac{1-\theta_n^*}{2-\theta_n^*}$ or $\theta_n^* = 1 - \frac{F_n^*}{1-F_n^*}$.

Note that $F_n = 0 (F_n^* = 0)$ gives $\theta_n^* = 1$ (diagonal case) and $F_n \geq 1/2 (F_n^* = 1/2)$ gives $\theta_n^* = 0$ (independence case).

Once θ_n^* is known we must estimate λ and δ. A natural choice is the ergodic theorem thus giving one equation

$\overline{X}_n = \lambda_n^* + \gamma\, \delta_n^*$.

For a second equation, we could use either

$\min(X_k - X_{k-1}) = \delta_n^* \log \theta_n^*$

(as $\min(X_k - X_{k-1}) = \delta \min(Z_k - Z_{k-1}) \stackrel{P}{\to} \delta \log \theta$)

or

$\max(X_1, \dots, X_n) = \lambda_n^* + \gamma + \log(\theta_n^* + n(1-\theta_n^*))\delta_n^*$.

As the use of the first relation, for δ_n^*, imposes one more condition (i.e., $\min(X_k - X_{k-1}) < 0$), we will use the second relation, with the ergodic theorem, to estimate λ_n^* and δ_n^*.

Thus we have the system

$\theta_n^* = 1 - \frac{F_n^*}{1-F_n^*}$

$\delta_n^* = \frac{\overset{n}{\underset{1}{\max}} (X_i) - \overline{X}_n}{\log(\theta_n^* + n(1-\theta_n^*))}$.

and $\lambda^* = \overline{X}_n - \gamma\, \delta_n^*$

to estimate the parameters. Note that the denominator of δ_n^* is always ≥ 0 for $n > 1$ and that $[\log(\theta_n^* + n(1-\theta_n^*))] / \log n \stackrel{P}{\to} 1$.

If we substitute the denominator of δ_n^* by $\log n$ we thus have the estimator $\delta_n^{**} = (\max_1^n(X_i) - \overline{X}_n)/\log n$ (independent of θ_n^*) and also the new estimator $\lambda_n^{**} = \overline{X}_n - \gamma\, \delta_n^{**}$. We will use these estimators.

As an example we will apply these simple estimators to two random sequences $\{E_j\}$ and $\{Z_j\}$ of 25 terms where $\{E_j\}$ is a sequence of independent reduced Gumbel random variables obtained from Goldstein (1963), and $\{Z_j\}$ is an EMS sequence with $\theta = 1/2$, i.e.,

$Z_k = \max(Z_{k-1}, E_k) - \log 2$:

Table 15.1

j	1	2	3	4	5	6	7
E_j	1.412	-0.296	-0.031	1.388	1.657	-0.382	-0.175
Z_j	1.412	0.719	0.026	0.695	0.964	0.271	-0.422

j	8	9	10	11	12	13	14
E_j	-0.380	-0.692	2.135	0.130	0.040	1.927	1.456
Z_j	-1.073	-1.385	1.442	0.749	0.056	1.234	0.763

j	15	16	17	18	19	20	21
E_j	0.472	1.140	-0.930	-0.793	-0.913	-0.610	0.616
Z_j	0.070	0.447	-0.246	-0.939	-1.606	-1.303	-0.077

j	22	23	24	25			
E_j	-0.920	1.434	1.586	3.688			
Z_j	-0.770	0.741	0.893	2.995			

Note that in the two cases we have $\lambda = 0, \delta = 1$. Assuming this, for the $\{E_j\}$ sequence we have $K_{25} = 13$ so $F_{25} = 13/24 (\geq 1/2)$ and so $\theta_{25}^* = 0$ (independence); for the $\{Z_j\}$sequence we have $K_{25} = 10$ and so $F_{25} = 10/24$, $F_{25}^* = F_{25}$ and $\theta_{25}^* = 2/7 = .286$ which is a long way from $\theta = 1/2$!

In the general case we can add the estimators of λ and δ.

For the $\{E_j\}$ sequence we have $\max_1^{25}(E_j) = 3.688, \overline{E}_{25} = .51716$ and so

$\delta_{25}^{**} = .985 \;\; \lambda_{25}^{**} = -.051,$

which are not very far from the exact values. Clearly, assuming $\theta = 0$, we should have used habitual ML estimators.

Consider, now, the $\{Z_j\}$ sequence. We have estimated, before, θ by $\theta_{25}^{*} = 2/7 = .286$.

As we have $\max_1^{25}(X_j) = 2.995$ and $\overline{X}_{25} = .226$ we get $\delta_{25}^{**} = .860$ and $\lambda_{25}^{**} = -.271$.

Once more the estimates are not close to the exact values!

The estimation problem has to be reconsidered.

15.3 Extreme-Markovian stationary (EMS) processes

After the definition of extremal processes and EMS sequences, we will define the EMS processes and relate them to the extremal processes, consider the associated maximum process, and show that its asymptotic behaviour is similar to that of an extremal process. This is analogous to the relation between the Wiener-Levy process and the integrated Orstein-Uhlenbeck process, as could be expected, to a certain extent, from the "duality" between maxima and sums.

In this section we will follow Tiago de Oliveira (1973).

An extreme-Markovian-stationary (EMS) process $Z(t)$ can be characterized by means of the following axioms:

- $Z(t)$ is a stationary process defined for $t(-\infty < t < +\infty)$;

- for $s \leq t, Z(t) = \max(Z(s) + \varphi(t-s), E(s,t) + \Psi(t-s)), Z(s)$ and $E(s,t)$ being independent[1];
- the random variables $E(s,t)$ have a reduced Gumbel distribution;
- $E(s,t)$ and $E(s',t')$ are independent if $]s,t[\cap]s',t'[=\emptyset$;
- $Z(0)$ is a reduced Gumbel random variable.

We will now deduce some basic results from the axioms.

From the stationarity we get

$\mathrm{Prob}(Z(t) \leq x) = \mathrm{Prob}(Z(0) \leq x) = \Lambda(x)$.

Now using the second axiom we get

$\Lambda(x) = \mathrm{Prob}(Z(t) \leq x) = \Lambda(x - \varphi(t-s))\,\Lambda(x - \Psi(t-s))$ so that

$e^{\varphi(t-s)} + e^{\Psi(t-s)} = 1$.

Now taking $s \leq u \leq t$ we get

$$Z(t) = \max(Z(s) + \varphi(t-s), E(s,t) + \Psi(t-s))$$

$$= \max(Z(s) + \varphi(u-s) + \varphi(t-u), E(s,u) + \Psi(u-s) + \varphi(t-u), E(u,t) + \Psi(t-u))$$

so that

$$\varphi(t-s) = \varphi(t-u) + \varphi(u-s)$$

$$E(s,t) + \Psi(t-s) = \max(E(s,u) + \Psi(u-s) + \varphi(t-u), E(u,t) + \Psi(t-u)).$$

As $e^{\varphi(t-s)} \leq 1$ the first relation gives, as $\varphi(w) \leq 0$,

$$\varphi(w) = -\beta\, w (\beta \geq 0)$$

so that

$$\Psi(w) = \log(1 - e^{-\beta\, w})$$

and, consequently, we get for $s \leq t, Z(t) = \max(Z(s) - \beta(t-s), E(s,t) + \log(1 - e^{-\beta(t-s)}))$.

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1. We could instead of this formulation introduce the random variables $E'(s,t) - E(s,t) + \Psi(t-s)$ with a Gumbel distribution (not reduced), adopting conveniently the axious and the proofs. This is left as an exercise.

Note that the random variables $E(s,t)$ satisfy the relation

$$E(s,t) = \max(E(s,u) + \log\frac{e^{-\beta s} - e^{-\beta u}}{e^{-\beta s} - e^{-\beta t}}, E(u,t) + \log\frac{e^{\beta t} - e^{\beta u}}{e^{\beta t} - e^{\beta s}})$$

for $0 < s \le u \le t$. It is immediate that

$$Z(t) = Z_0(e^{\beta t}) - \beta t,$$

where $Z_0(t')$ is a (reduced) extremal process.

The joint distribution function of $(Z(t_1), \dots, Z(t_n))$ is easily shown to be

$$\Lambda_n(x_1, t_1; \dots; x_n, t_n) = \text{Prob}\{Z(t) \le x_1, \dots, Z(t_n) \le x_n\}$$

$$= \Lambda_{n-1}(x_1, t_1; \dots; x_{n-2}, t_{n-2}; \min(x_{n-1}, x_n + \beta(t_n - t_{n-1})), t_{n-1}).$$

$$\cdot \Lambda(x_n - \log(1 - e^{-\beta(t_n - t_{n-1})})).$$

From the previous result we get

$$\Lambda_2(x_1, t_1; x_2, t_2) = \exp(-(e^{-x_1} + e^{-x_2}) - \min(e^{-x_1}, e^{-\beta(t_2 - t_1)} e^{-x_2}))$$

so that the dependence function is

$$k(w) = 1 - \max(e^{-\beta(t_2 - t_1)}, e^w)/(1 + e^w),$$

a biextremal one with the parameter $\theta = e^{-\beta(t_2 - t_1)}$.

The correlation coefficient is then

$$\frac{6}{\pi^2} R(e^{-\beta(t_2 - t_1)}),$$

now being continuous in the diagonal.

Also the conditional distribution function of the EMS process is

$$\text{Prob}(Z(t) \le y | Z(s) = x) = 0 \quad \text{if } y < x - \beta(t - s)$$

$$= \Lambda(y - \log(1 - e^{-\beta(t-s)})) \quad \text{if } y \ge x - \beta(t - s),$$

with a jump of

$$\Lambda(x - \log(e^{\beta(t-s)} - 1)) \quad \text{at } y = x - \beta(t - s),$$

for $s \le t$.

Because $Z(t)$ is a Markovian process, the least squares predictor of $Z(t)$ when $Z(t_1) = x_1, \ldots, Z(t_n) = x_n (0 < t_1 < t_2 \ldots < t_n < t)$ is the conditional mean value of $Z(t)$ when $Z(t_n) = x_n$,

$$p(t; x_n, t_n) = x_n - \beta(t - t_n) + \int_{x_n}^{+\infty} [1 - \Lambda(x - \log(e^{\beta(t-t_n)} - 1))] d\,x$$

and the conditional mean-square error is, after a simple algebra,

$$\int_{x_n}^{+\infty} [1 - \Lambda(x - \log(e^{\beta(t-t_n)} - 1))] d\,x^2 - [(p(t; x_n, t_n) + \beta(t - t_n))^2 - x_n^2].$$

Recall that also for the EMS process, from the stationarity, we have the mean value γ and the variance $\pi^2/6$.

Let us now study $\tilde{Z}(t) = \max_{0 \le s \le t} Z(s), (0 \le s \le t)$. The definition has effective meaning because of the relationship between $Z(t)$ and $Z_0(t)$. The correlation coefficient being continuous on the diagonal, Theorem C in Loéve (1961), shows that $Z(t)$ has many separability sets, one of them being the set of non-negative rationals.

We can then compute $\text{Prob}\{\tilde{Z}(t) \le x\}$. Fixing $h > 0$, for

$$F_n(x) = \text{Prob}\{\bigcap_0^n \{Z(k\,h) \le x\}\}$$

we have

$$F_n(x) = \text{Prob}\{\bigcap_0^{n-1} \{Z(k\,h) \le x\} \cap (\max(Z(n-1)h) - \beta\,h, E((n-1)\,h, n\,h) + \log(1 - e^{\beta h})) \le x\}\}$$

$$= F_{n-1}(x) \cdot \exp\{-(1 - e^{-\beta\,h})e^{-x}\}$$

$$= F_0(x) \cdot \exp(-n(1 - e^{-\beta\,h})e^{-x}) = \exp\{-(1 + n(1 - e^{-\beta\,h}))e^{-x}\}.$$

Taking now $\{h_n\}$ rational and such that $n\,h_n \to t$, we get

$$\text{Prob}\{\tilde{Z}(t) \le x\} = \exp(-(1 + \beta\,t)e^{-x}),$$

so that $\tilde{Z}(t) - \log(1 + \beta\,t)$ is a reduced Gumbel random variable.

Analogously we can show, for $s \le t$, that we have

$$\text{Prob}\{\tilde{Z}(s) x, \tilde{Z}(t) \le y\} = \exp\{-(1 + \beta\,s)\max(e^{-x}, e^{-y}) - \beta(t - s)e^{-y}\}.$$

The $\tilde{Z}(t)$ process is evolutionary. Using the reduced margins $\xi = x - \log(1+\beta\, s), \eta = y - \log(1+\beta\, t)$ we get the dependence function

$$k(w) = 1 - \min(\theta, e^w)/(1+e^w) \qquad \text{with} \qquad \theta = \frac{1+\beta\, s}{1+\beta\, t}.$$

For large values of s and t we have $\theta \simeq s/t$ which suggests that $Z(t)$ is asymptotically similar to the external process.

Note that the mean value function is $\bar{\mu}(t) = \gamma + \log(1+\beta\, t)$, the variance is constant $(\pi^2/6)$ and the correlation coefficient is $\frac{6}{\pi^2} R(\frac{1+\beta\, s}{1+\beta\, t})$ for $s \le t$.

For the times $\tau_i = \log(1+\beta\, t_i)$ we get

$$\text{Prob}\{Z(t_i) \le x_i, 0 < t_1 < t_2 < \cdots < t_n\} = \prod_{i=1}^{n} \Lambda(\min(x_i, \dots, x_n) - (\tau_i - \tau_{i-1}))$$

so that, for the new timing, $\tilde{Z}(\tau) = \tilde{Z}((e^\tau - 1)/\beta)$ is exactly an external process, and for large t_i and $\beta = 1$ (change of the time unit) we have $\theta_i \sim t_i$ so that $\tilde{Z}(t)$ is asymptotically an extremal process.

We see immediately that $Z(t)/\log(1+\beta\, t) \overset{ms}{\rightarrow}$ as $t \to \infty$.

For small values of $s \le t\, (\beta\, t \ll 1)$ we have $\frac{1+\beta\, s}{1+\beta\, t} \sim 1$ and $\tilde{Z}(t) - \log(1+\beta\, t) \simeq \tilde{Z}(t) - \beta\, t$ is approximately a reduced Gumbel random variable; the process is stationary only to the first order.

The relationship between $Z(t)$ and $\tilde{Z}(t)$ has a complex joint behaviour of $Z(s)$ and $\tilde{Z}(t)$, as the bivariate distribution is, unexpectedly, not biextremal.

By the technique previously used, we can show that

$$\Psi(x, y|t) = \text{Prob}(Z(t) \le x, \tilde{Z}(t) \le y) = \exp\{-\max(e^{-x-\beta\, t}, e^{-y})$$

$$- \int_0^{\beta\, t} \max(e^{-h-x}, e^{-y}) d\, h\},$$

the margins being $\exp(-e^{-x})$ and $\exp(-(1+\beta\, t)\, e^{-y})$. We have $\Psi(x, y|t) = \Psi(\min(x, y), y|t)$ from the definition. Then the dependence function, for the reduced margins, is

$$k(w) = \frac{1}{1+e^{-w}} \{\max(e^{-\beta t}, \frac{e^{-w}}{1+\beta\, t} + \int_0^{\beta\, t}) \max(e^{-h}, \frac{e^{-w}}{1+\beta\, t}) \, d\, h\},$$

clearly not a biextremal one as shown in the more detailed form for $k(w)$

$$k(w) = \frac{1}{1+e^{w}} \quad \text{if} \quad w < -\log(1+\beta\, t)$$

$$= 1 - \frac{1+w+\log(1+\beta\, t)}{(1+\beta\, t)(1+e^{w})} \quad \text{if} \quad -\log(1+\beta\, t) \le w \le \beta\, t - \log(1+\beta\, t)$$

$$= \frac{e^{w}}{1+e^{w}} \quad \text{if} \quad w > \beta\, t - \log(1+\beta\, t);$$

but it should be noted that this $k(w)$ is a generalized form of the biextremal dependence function, the natural one.

It is also easy to compute

$$G(w|t) = \mathrm{Prob}\{\tilde{Z}(t) - Z(t) \le w\}.$$

Evidently we must have $G(w|t) = 0$ for $w < 0$ and for $t = 0$ we have $G(w|0) = H(w)$.

The general expression for $t > 0$ is

$$G(w|t) = \frac{e^{-\beta\, t} H(w-\beta\, t) + \int_0^{\beta\, t} e^{-h} H(w-h)\, d\, h}{\max(e^{-\beta t}, e^{-w}) + \int_0^{\beta\, t} \max(e^{-h}, e^{-w})\, d\, h}.$$

This result shows, as $G(w|t) = 1$ for $w > \beta\, t$, that, with probability one,

$$\tilde{Z}(t) \le Z(t) + \beta\, t.$$

We can show this result directly, as follows.

From the basic equation we get $Z(t) \ge Z(s) - \beta(t-s)$ with $s \le t$, and as $\tilde{Z}(t) = Z(s_1)$ we have

$$Z(t) \ge Z(s_1) - \beta(t - s_1)$$

and thus

$$\tilde{Z}(t) \le Z(t) + \beta(t - s_1) \le Z(t) + \beta\, t.$$

The correlation coefficient between $Z(t)$ and $\tilde{Z}(t)$, as follows from the natural model, is

$$\rho(t) = 1 + 6/\pi^2 \Big[\frac{\beta\, t - \log(1+\beta\, t)^2}{2} - \int_0^{\beta\, t} \log(e^{w} + \beta\, t - w)\, d\, w\Big] = 1 - 6/\pi^2 \cdot \beta^2\, t^2 + \cdots$$

The statistical decision for an EMS process has not yet been considered. Only a few suggestions will be made here.

Two ways can be used to approach the estimation of the parameters of the general stochastic process $X(t) = \lambda + \delta\, Z(t)$, with $Z(t)$ a reduced EMS process.

One way is to consider the sequence $X_j = X(t_0 + (j-1)\, h)$ observing $X(t)$ at equal time steps and considering it as an EMS sequence. In that case we have for the parameters θ of the EMS sequence and β of the EMS process the relation $\theta = e^{-\beta\, h}$, and then we estimate θ (or β), λ, and δ as in the previous section.

Another way is to recall that if $Z(t)$ is an EMS process then $Z(\frac{\log t}{\beta}) + \log t$ is an extremal process for $t \geq 0$ and thus, once β is estimated, we can estimate λ and δ for extremal processes, as before.

15.4 Extreme-Markovian-evolutionary (EME) sequences

Let $\{E_j\}(j = 0,1,2,\dots)$ be a sequence of i.i.d. random variables with standard Gumbel margins, and $X_0 = \lambda_0 + \delta_0\, Z_0$ a Gumbel random variable with parameters $(\lambda_0 + \delta_0)$, i.e., Z_0 is a standard Gumbel random variable; Z_0 is independent of all $\{E_j\}$.

In this section and the next we will follow Tiago de Oliveira (1986). Let us consider the auto-regressive sequence

$$X^*_{j+1} = \max(a + b\, X_j, a' + b'\, E_j), j = 0,1,2, \dots$$

where the X^*_j are assumed to be Gumbel random variables. Let (λ_j, δ_j) be the parameters of X_j , i.e., $X_j = \lambda_j + \delta_j\, Z_j$ with Z_j reduced Gumbel random variables. The auto-regressive relation can then be written as

$$\lambda_{j+1} + \delta_{j+1}\, Z_{j+1} = \max\big(a + b\big(\lambda_j + \delta_j\, Z_j\big), a' + b' E_j\big), j = 0,1,2, \dots$$

Thus $\text{Prob}\{Z_{j+1} \leq x\}$ is $\Lambda(x)$ if

$$\delta_{j+1} = b\, \delta_j$$

$$\delta_{j+1} = b'$$

$$e^{a + b\,(\lambda_j - \lambda_{j+1})/b\,\delta_j} + e^{(a' - \lambda_{j+1})/b'} = 1$$

so that $b' = \delta_j = \delta_0, b = 1$ and $e^{\lambda_{j+1}/\delta_0} = e^{(a+\lambda_j)/\delta_0} + e^{a'/\delta_0}$, and the auto-regressive relation takes the simpler form

$$X_{j+1} = \max(a + X_j, + a' + \delta_0 E_j).$$

If we introduce, for convenience, the "patterned" sequence $Y_j = (X_j - \lambda_0)/\delta_0 (Y_0 = Z_0)$ with $Z_j = (X_j - \lambda_j)/\delta_j = Y_j - (\lambda_j - \lambda_0)/\delta_0 = Y_j - \eta_j, (\eta_j = (\lambda_j - \lambda_0)/\delta_0, \eta_0 = 0)$. Y_j then satisfies the auto-regressive equation

$$Y_{j+1} = \max(a_0 + Y_j, a'_0 + E_j), j = 0, 1, 2, \ldots$$

with $a_0 = a/\delta_0, a'_0 = (a' - \lambda_0)/\delta_0$, and the relation for $\{\lambda_j\}$ takes the form

$$e^{\eta_{j+1}} = e^{a_0} e^{\eta_j} + e^{a'_0}.$$

In brief, the EME-sequence $\{X_j\}$, with X_0 with parameters (λ_0, δ_0), satisfies the relation

$$X_{j+1} = \max(a_0 \delta_0 + X_j, \lambda_0 + a'_0 \delta_0 + \delta_0 E_j)$$

and the "patterned" sequence verifies

$$Y_{j+1} = \max(a_0 + Y_j, a'_0 + E_j);$$

(a_0, a'_0) are then the *essential* parameters and (λ_0, δ_0) are *incidental* parameters; we can reconstitute X_j by the relation $X_j = \lambda_0 + \delta_0 Y_j$, and as $Y_j = Z_j + \eta_j$ the margin parameters of X_j are $(\lambda_0 + \delta_0 \eta_j, \delta_0)$.

Consider, now, the difference equation

$$e^{\eta_j+1} = e^{a_0} e^{\eta_j} + e^{a'_0}, \text{with } \eta_0 = 0:$$

if $e^{a_0} = 1 (a_0 = 0)$ we have $e^{\eta_j} = 1 + e^{a'_0} j \; (\eta_j = \log(1 + e^{a'_0} j) > 0)$;

if $e^{a_0} \neq 1 \; (a_0 \neq 0)$ we obtain

$$e^{\eta_j} = \frac{e^{a'_0}}{1 - e^{a_0}} + (1 - \frac{e^{a'_0}}{1 - e^{a_0}}) e^{a_0 j} = A + (1 - A) e^{a_0 j} (\geq 0)$$

with $A = \frac{e^{a'_0}}{1 - e^{a_0}}$; note that $\lim_{a_0 \to 1} e^{\eta_j} = 1 + e^{a'_0} j$, the expression of e^{η_j} for $a_0 = 0$.

The condition of stationarity imposes $\eta_{j+1} = \eta_j = \cdots = \eta_0 = 0$ so that $e^{a_0} + e^{a'_0} = 1$; this is the condition for stationarity obtained previously in EMS sequences; $\theta = e^{a_0}$ was there a dependence (essential) parameter.

As we have only fragmented results we will study some detailed features of the EME sequence.

Let us consider the monotonicity behaviour of the "patterned" sequence. As $M(Y_j) = M(\eta_j + Z_j) = \gamma + \eta_j$; we see that Y_j (and thus X_j) are increasing, constant or decreasing in mean according to $\eta_{j+1} > \eta_j, \eta_{j+1} = \eta_j$ or $\eta_{j+1} < \eta_j$, i.e., according to $e^{a_0} + e^{a'_0} > 1, e^{a_0} + e^{a'_0} = 1$ (stationarity) or $e^{a_0} + e^{a'_0} < 1$.

If $a_0 = 0$ we do not have the decreasing behaviour and we get constancy only if $a_0 = 0$ and $a'_0 = -\infty$, i.e., $Y_{j+1} = Y_j$.

Now compute $\text{Prob}\{Y_{j+1} = Y_j\}$. If $a_0 > 0$ it is immediate that $Y_{j+1} > a_0 + Y_j$ so that $\text{Prob}\{Y_{j+1} > Y_j\} = 1$: the EME sequence is increasing with probability one, and so the method of "ups and downs" considered before for EMS sequences for the estimation of $\theta = e^{a_0}$, which should be < 1, cannot be used.

For $a_0 = 0$ as $Y_{j+1} \geq Y_j$ we have $\text{Prob}\{Y_{j+1} > Y_j\} = 1/(1 + a'_0 + j)$ decreasing with j and as $\text{Prob}\{Y_{j+1} > Y_j\} = 1 - \text{Prob}\{Y_{j+1} > Y_j\} \to 1$ we see that the sequence stabilizes asymptotically.

When $a_0 < 0$ we have $\text{Prob}\{Y_{j+1} > Y_j\} = 1 - \text{Prob}\{Y_{j+1} \leq Y_j\} = (1 - e^{a_0})\, e^{a'_0}/[(2 - e^{a_0})e^{a'_0} + (1 - e^{a_0} - e^{a'_0})\, e^{a'_0 j}]$ with value $e^{a'_0}/(1 + e^{a'_0})$ at $j = 0$, converging to $(1 - e^{a_0})/(2 - e^{a_0})$ as $j \to \infty$ increases if $e^{a_0} + e^{a'_0} < 1$, behaving in a stable way if $e^{a_0} + e^{a'_0} = 1$ (constancy), and decreasing if $e^{a_0} + e^{a'_0} > 1$.

The "patterned" sequence $\{Y_j\}$ and also $\{X_j\}$ increase in median $\text{Prob}\{Y_{j+1} \geq Y_j\} > 1/2$ always if $a_0 > 0$, when $j \geq 1 - e^{a'_0}$ if $a_0 = 0$ and $a'_0 > 0$, but never when $a'_0 < 0$, and when $e^{-a_0 j} < (e^{a_0} + e^{a'_0} - 1)e^{-a_0} + e^{-a'_0}$ if $e^{a_0} + e^{a'_0} > 1$ but never if $e^{a_0} + e^{a'_0} \leq 1$.

We will obtain the bivariate structure of an EME sequence, the multivariate structure being an immediate extension.

Taking $i < j$ we have

$$Y_j = \max(a_0 + Y_{j-1}, a'_0 + E_{j-1}) = \max(2a_0 + Y_{j-2}, a_0 + a'_0 + E_{j-2}, a'_0 + E_{j-1}) = \cdots$$

$$= \max((j-i)a_0 + Y_i, a'_0 + \max_{p=1}^{j-1}((p-1)a_0 + E_{j-p})).$$

Then $\text{Prob}\{Y_i \le x, Y_j \le y\} = \Lambda(\min(x, y-(j-i)a_0 - \eta_i))$

$$\times \prod_{p=1}^{j-i} \Lambda(y - a'_0 - (p-1)a_0) \text{ if } a_0 \neq 0$$

and

$$\text{Prob}\{Y_i \le x, Y_j \le y\} = \exp\{-\max(e^{-x}, e^{-y})e^{\eta_i} - a'_0(j-i)e^{-y}\}$$

if $a_0 = 0$.

The dependence function associated with the bivariate structure of $(Z_i, Z_j) = (Y_i - \eta_i, Y_j - \eta_j)$ is

$$\text{Prob}\{Z_i \le x, Z_j \le y\} = \text{Prob}\{Y_i \le x + \eta_j, Y_j \le y + \eta_j\} = (\Lambda(x)\,\Lambda(y))^{k_{i,j}(y-x)},$$

where

$$k_{i,j}(w) = 1 - \frac{\min(e^{(j-i)a_0 - \eta_j + \eta_i}, e^w)}{1+e^w}.$$

The correlation coefficient is, then,

$$\rho_{i,j} = \frac{6}{\pi^2} R(e^{(j-i)a_0 - \eta_j + \eta_i}).$$

where $0 \le e^{(j-i)a_0 - \eta_j + \eta_i} \le 1$. For $a_0 = 0$ we get $\rho_{i,j} = \frac{6}{\pi^2} R(\frac{1+e^{a'_0 i}}{1+e^{a'_0 j}})$ and for $a_0 \neq 0$

We have

$$\rho_{i,j} = \frac{6}{\pi^2} R \frac{1 - A + A\,e^{a'_0 i}}{1 - A + A\,e^{a'_0 j}},$$

where the argument of $R(.)$ is between 0 and 1. For any EMS sequence, with $e^{a_0} + e^{a'_0} = 1$ we get $\rho_{i,j} = \frac{6}{\pi^2} R(e^{a_0(j-i)})$ evidently with $a_0 < 0$ as obtained previously, with $a_0 = \log\theta, a'_0 = \log(1-\theta), 0 \le \theta \le 1$: independence $(\theta = 0)$ gives $\rho_{i,j} = 0$.

Let us obtain some more propositions that can be useful for the statistical analysis of the EME sequences.

A first result is that $Y_j/j \xrightarrow{ms} \max(a_0, 0)$ and $Y_{j+1} - Y_j \xrightarrow{ms} a_0$, if $a_0 > 0$, as $j \to \infty$. From $Y_j = \eta_j + Z_j$ we obtain

$M(Y_j/j) = (\gamma + \eta_j)/j \to \max(a_0, 0)\, , V(Y_j/j) = (\pi^2/6)/j^2 \to 0$ and

$M(Y_{j+1} - Y_j) = \eta_{j+1} - \eta_j \to a_0$ if > 0 and $V(Y_{j+1} - Y_j) = \pi^2/3(1 - \rho_{i,j+1}) \to 0$.

As $Y_{j+1} - Y_j = a_0 (\text{or } a'_0 + E_j \le a_0 + Y_j)$ with probability

$\frac{e^{a_0} \cdot e^{\eta_j}}{e^{a'_0} + e^{a_0} e^{\eta_j}} \to e^{\min(a_0,0)}$ as $j \to \infty$ and

$Y_{j+1} - Y_j > a_0$ with the complementary probability, it seems natural to study the statistic $A_n = \min_1^n (Y_j - Y_{j-1})$.

Let $Q_n(\Delta_1, \dots, \Delta_n)$ denote $\text{Prob}\{Y_1 - Y_0 > a_0 + \Delta_1, \dots, Y_n - Y_{n-1} > a_0 + \Delta_n\}$ for $\Delta_i \ge 0$. It is immediate that the event $D_n = \{Y_1 - Y_0 > a_0 + \Delta_1, \dots, Y_n - Y_{n-1} > a_0 + \Delta_n\}$, as $Y_j = a'_0 + E_j$, is equivalent to $D'_n = \{a'_0 + E_1 > a_0 + \Delta_1 + Y_0, E_2 > a_0 + \Delta_2 + E_1, \dots, E_n > a_0 + \Delta_n + E_{n-1}\}$, and we get

$$Q_n(\Delta_1, \dots, \Delta_n) = G_n(e^{a_0 + \Delta_1 - a'_0}, e^{a_0 + \Delta_2}, \dots, e^{a_0 + \Delta_n})$$

where

$$G_n(\varphi_1, \dots, \varphi_n) = \int_{\substack{t_0 > \varphi_1 t_1 \\ \cdots \\ t_{n-1} > \varphi_n t_n}} e^{-(t_0 + t_1 + \cdots + t_n)} d\,t_0\ d\,t_1 \dots d\,t_n,$$

which satisfies the relation

$$G_n(\varphi_1, \dots, \varphi_n) = \frac{1}{1 + \varphi_1} G_{n-1}((1 + \varphi_1)\varphi_2, \varphi_3, \dots, \varphi_n).$$

Then

$$Q_n(\Delta,\dots,\Delta) = \frac{1}{1+e^{a_0+\Delta-a'_0}}\, Q_{n-1}((1+e^{a_0+\Delta-a'_0})e^{a_0+\Delta},\dots,e^{a_0+\Delta})$$

tends to 0 as $n \to \infty$. A_n is thus an estimator of a_0.

Some other results give hints for statistical estimation. It is easily shown that, as

$M(Y_{j+1} - Y_j - a_0) \overset{ms}{\rightarrow} -\min(a_0,0)\,, \sum_0^{n-1} M(Y_{j+1} - Y_j - a_0) = Y_n - Y_0 - n\,a_0$

$\rightarrow \log(1 - \frac{e^{a_0}}{1-e^{a_0}})$ if $a_0 > 0$ and $\sum_0^{n-1} M(Y_{j+1} - Y_j) \rightarrow a'_0 - \log(1-e^{a_0})$ if $a_0 > 0$, as $n \rightarrow \infty$.

We have also, for $a_0 \le 0$, $\mathrm{Prob}\{\max(Y_0, Y_1, \dots, Y_n) \le x\}$

$= \mathrm{Prob}\{\max(Y_0, a'_0 + E_0, \dots, a'_0 + E_{n-1}) \le x\} = \Lambda(x)\,\Lambda^{n-1}(x - a'_0)$

$= \exp\{-e^{-x}(1 + (n-1)e^{a'_0})\}$

and so $\mathrm{Prob}\{\max(Y_0, Y_1, \dots, Y_n) - \log n \le x\} = \Lambda(x - a'_0)$.

15.5 Some remarks on statistical decision for EME sequences

We have not obtained sufficient results even for simple and quick statistical decision but some remarks can be made.

It is natural, in this initial phase, to divide statistical decision for the EME sequences into two steps: statistical decision concerning the (essential) parameters (a_0, a'_0) and then, supposing (a_0, a'_0) is known, to estimate (λ_0, δ_0) by the least squares method. In principle, the estimators of (a_0, a'_0) must be independent of (λ_0, δ_0) and those of the incidental parameters must be quasi-linear, i.e., such that $\lambda_0^*(\alpha + \beta\, X_j) = \alpha + \beta\,\lambda_0^*(X_j)$ and $\delta^*(\alpha + \beta\, X_j) = \beta\,\delta_0^*(X_j)$ for $-\infty < \alpha < +\infty, 0 < \beta < +\infty$, as happens with the least squares method; see Cramér (1946) and Silvey (1975).

Let us now suppose we are dealing with the "patterned" sequence $\{Y_j\}$.

A test of constancy $(a_0 = 0, a'_0 = -\infty)$ is not necessary; to devise tests of independence $(a_0 = -\infty, a'_0 = 0)$ and of stationarity is very important. We will consider only the important case where $a_0 > 0$.

As $Y_j/j \overset{ms}{\rightarrow} a_0 (> 0)$, a natural region for deciding $a_0 > 0$ is to accept this hypothesis if $\{X_j > A_j\}$, which is also the Neyman-Pearson test of $\lambda > 0$

against $\lambda \leq 0$ for the distribution $\Lambda(x-\lambda)$. Recall that $\eta_j - a_0 \to \log(1 + \frac{e^{a'_0}}{e^{a_0}-1})$ if $a_0 > 0, \eta_j - \log j \to a'_0$ if $a_0 = 0$, and $\eta_j \to a'_0 - \log(1 - e^{a_0})$ if $a_0 < 0$ as $j \to \infty$; η_j increases linearly with j if $a_0 > 0$, logarithmically if $a_0 = 0$, and converges to a constant if $a_0 < 0$.

A_j can be defined by imposing $\text{Prob}\{Y_j \leq A_j | a_0 = 0\} = \alpha$ or $\Lambda(A_j - \log(1 + e^{a'_0} j)) = \alpha$ and so $A_j = \log(1 + e^{a'_0} j) - \log(-\log \alpha)$ still depending on the fixation of the value a'_0.

A last remark: when $a_0 > 0$, as $Y_{j+1} \geq a_0 + Y_j$ we have $Y_j \geq Y_0 + a_0\, j$, so that after some steps (depending on the random Y_0 and on (a_0, a'_0)) we will practically always have $Y_{t+1} = a_0 + Y_t$ because $\text{Prob}(a'_0 + E_j > Y_0 + a_0\, j) = \frac{e^{a'_0}}{e^{a'_0} + e^{a_0} j} \to 0$ very quickly. In practice a'_0, can be estimated only by the first values of $\{Y_t\}$, if possible. The difficulty is analogous to the no-jump situation (non-increasingness) in extremal sequences/processes.

Statistical decision for EME-sequences is still open; the results given here may be helpful in some cases for a first step.

Evidently if we have, or suppose, or assume, that the above processes have margins that are not Gumbel but Fréchet or Weibull, by the usual transformations, estimating the convenient parameters, we can reduce them to Gumbel margins.

15.6 Sliding extreme (SE) sequences

Consider a (doubly) infinite sequence of *independent* random variables $\ldots X_{-1}, X_0, X_1, \ldots, X_n, \ldots$ which are assumed to have Gumbel distribution.

We will suppose that if the sequence is *stable* the X_i have the parameters (λ, δ), i.e.,

$\text{Prob}\{X_j \leq x\} = \Lambda((x-\lambda)/\delta)$,

but if the sequence is unstable the parameters are $(\lambda + (v + n\,\theta)\delta, \delta)$ (v and $\theta > 0$ unknown), i.e., the distribution function of the X_j is

$\Lambda(((x-\lambda) - (v + j\,\theta)\delta)/\delta)$.

We will obtain a test of $\theta = 0$ vs.$\theta > 0$, i.e. *stability* vs. (*positive) instability*.

Here and in the next section will follow Tiago de Oliveira (1987).

Evidently, if the distributions of the observations are Weibull or Fréchet, the usual log-transformations will reduce them to the present case, as is well known and will be done later.

Thus we can write

$X_j = \lambda + (v + j\,\theta)\delta + \delta\, Z_j$

where $\{Z_j\}$ are independent reduced Gumbel random variables with $v = 0$ and $\theta = 0$ in the stable case and $\theta > 0$ in the (positive) unstable case (linear increase of the location parameter).

The underlying idea is that, although we assume independence either in the stable or the unstable cases, independence will act as a reference pattern for significance tests.

The nuisance parameter $v > 0$ can be interpreted as meaning that instability began somewhere in the past ($j_0 < 0$), before sampling, or even can absorb a wrong choice of λ.

It is easy to show that if r_n is the correlation coefficient between $(1, 2, \ldots, n)$ and $(X_1, \ldots, X_n)$, $r_n \overset{P}{\to} 0$ if $\theta = 0$ and $r_n \overset{P}{\to} 1$ if $\theta > 0$.

Note that if we take $Y = \overline{w} - e^{-X}$, where X has the Gumbel distribution with parameters (λ, δ), then Y has the Weibull distribution

$\text{Prob}\{Y \le y\} = \exp\{-(\frac{\overline{w} - y}{e^{-\lambda}})^{1/\delta}\}$ for $y \le \overline{w}$

$= 1$ for $y \ge \overline{w}$

with the location parameter (right-end point) $\overline{w}$, the dispersion parameter $\tau = e^{-\lambda}$ – decreasing to zero if $\lambda \to +\infty$ and thus increasing the probability $\text{Prob}\{Y > y\} = 1 - \text{Prob}\{Y \le y\}$, which is relevant to earthquakes - and shape parameter $\alpha = 1/\delta$; if we take $Y = e^X + \underline{w}$, with X also a Gumbel random variable with parameters (λ, δ), then

$\text{Prob}\{Y \le y\} = 0$ if $y \le \underline{w}$

$= \exp\{-(\frac{y - \underline{w}}{e^{-\lambda}})^{-1/\delta}\}$ if $y \ge \underline{w}$

a Fréchet distribution with location parameter (left-end point) $\underline{w}$, dispersion parameter $\tau = e^{\lambda}$ - increasing with $\lambda \to \infty$ and thus seeming irrelevant for applications - and shape parameter $\alpha = 1/\delta$.

For earthquake applications Yegulalp and Kuo (1974) have shown that the Weibull distribution gives a good fit, but accepting the Gumbel distribution in some seismic areas; for the area considered in the case study Ramachandran (1980) says that the Gumbel distributions gives a good fit; for some remarks on global modelling for seismic areas see Tiago de Oliveira (1984).

15.7 Statistical decision for SE sequences

The likelihood of the sample $(x_1, \dots, x_n)$ is

$$L = L(x_1, \dots, x_n) = \frac{1}{\delta^n} \exp\{-\textstyle\sum_1^n (\frac{x_j-\lambda}{\delta} - v - j\,\theta)\}$$

$$\exp\{-\textstyle\sum_1^n e^{-((x_j-\lambda)/\delta - v - j\,\theta)}\},$$

and so the LMP test of $\theta = 0$ vs. $\theta > 0$ (v assumed to be zero) is given by the rejection region

$$\frac{\partial \log L}{\partial \theta}\Big|_{\theta=0} \geq c'_n$$

or

$$T_n = \textstyle\sum_1^n j \exp(-\frac{x_j-\lambda}{\delta}) \leq c_n.$$

In the tested stability $(v = 0, \theta = 0)$, the $\exp(-\frac{x_j-\lambda}{\delta}) = E_j$ being standard exponential random variables, the distribution of T_n is that of $\sum_1^n j\, E_j$, with the $\{E_j\}$ independent.

It is obvious that if

$$F_n(c) = \text{Prob}\{\textstyle\sum_1^n j\ E_j \leq c\}$$

we have

$$F_n(c) \leq F_{n-1}(c) \leq \cdots \leq F_1(c) = 1 - e^{-c}$$

with

$$F_n(c) \leq F_{n-1}(c) - e^{-c/n} \int_0^c e^{y/n}\, d\, F_{n-1}(y),$$

and so, for instance,

$F_2(c) = 1 + e^{-c} - 2\, e^{-c/2}$.

Denoting by $c_n(\alpha)$ the solution of $F_n(x) = 1 - \alpha$, we see that $c_n(\alpha) > c_{n-1}(\alpha)$.

We have

$c_1(\alpha) = -\log\alpha\,, c_2(\alpha) = -2\log(1-\sqrt{1-\alpha})$

which for $\alpha = .25$, as

$c_1(.05) = 2.9957323 < c_2(.05) = 7.3522767$,

gives an idea of the initial rate of increase of $c_n(.05)$.

If $v \neq 0$ and $\theta = 0$ (instability *before* the sample) the correct statistic would be $T_n e^{-v}$ (with T_n as before), the correct region would be $T_n e^{-v} \le c_n$ smaller than $T_n \le c_n$, thus giving over-rejection of stability.

T_n has the mean value $n(n+1)/2$, variance $n(n+1)\,(2\,n+1)/6$, and we can show easily that

$T'_n = \frac{T_n - n(n+1)/2}{\sqrt{n(n+1)\,(2\,n+1)/6}}$

is asymptotically standard normal; but this result is not very useful because we will deal with small values of n in applications.

The test assumes (λ, δ) known (stable case); we can presume it in some cases, such as earthquakes, because for each seismic region the long history gives sufficiently good estimates of the parameters to be used.

If we were dealing with the Weibull distribution, with location dispersion $\overline{w}$, dispersion parameter τ, and shape parameter α, the statistic T_n takes the form

$T_n = \sum_1^n j\,(\frac{\overline{w}-y_j}{\tau})^{\alpha}$;

when dealing with the Fréchet distribution with location parameter $\underline{w}$, dispersion parameter τ, and shape parameter α, the statistic T_n is

$T_n = \sum_1^n j\,(\frac{y_j-\underline{w}}{\tau})^{-\alpha}$.

The distribution of T_n in all three cases is the same, in the stable case.

Let us now consider a case study, the waning down of the Santa Barbara earthquake of 13th August 1978.

The daily maximum magnitudes for 13th August and the following days until 27th August, with no observation at the 27th, are

Table 15.2

Day	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Mag.	5.1	3.1	2.4	2.7	2.6	2.3	2.3	3.1	2.6	2.1	2.3	2.4	2.1	2.7	--

as given in Corbett and Johnson (1982).

Taking the last six daily maxima of the paper, which goes to 30th September (2.0, 3.5, 1.9, 2.0, 2.0, 1.8), the Leiblein-Zellen estimators - see Chapter 5 for details - are $\lambda^* = 1.980$ and $\delta^* = .314$, and the estimates given by Ramachandran (1980) for the area (case c) are $\lambda^{**} = 1.825$ and $\delta^{**} = 1/3.425 = .292$. For simplicity we will take $\tilde{\lambda} = 2.0$ and $\tilde{\delta} = .3$. Using the first six observed daily maxima *after* the earthquake, in *reverse order* to take account of the expected downward trend, we get

$$T_n = \sum_1^6 j\exp\left(-\frac{x_j-2}{.3}\right) = 3.3688867 < c_2(0.05);$$

we must conclude that a downward trend exists. The closeness of the values of (λ^*, δ^*) and $(\lambda^{**}, \delta^{**})$ can be interpreted as meaning that after, approximately, two weeks the usual stability was practically attained.

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