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Part 1
Probabilistic Patterns of Univariate Statistical Extremes

Annex 3

On the δ-Method

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Essentially the *δ-method* is based on the fact that if $\{X_k\}$ and $\{Y_k\}$ are sequences of random variables such that either $X_k - Y_k \xrightarrow{P} 0$ or $X_k/Y_k \xrightarrow{P} 1$ and $\{X_k\}$ has a limiting proper and non-degenerate distribution, the same happens to $\{Y_k\}$; see Cramér (1946). Also if $\{(X_k, Z_k)\}$ and $\{(Y_k, Z_k)\}$ are sequences of random pairs and $X_k - Y_k \xrightarrow{P} 0$ or $X_k/Y_k \xrightarrow{P} 1$ and $\{(X_k, Z_k)\}$ has a limiting (bivariate) proper and non-degenerate distribution, the same happens to $\{(Y_k, Z_k)\}$. The multivariate generalization is obvious. In many applications the δ-method leads to a linearization (use of the terms up to the first order of Taylor development). For details see Tiago de Oliveira (1982).

The δ-method was underlying some reasoning in this Part and will be used more extensively in the following Parts.

Let us give two examples, used in the text, that help to clarify the use of the δ-method.

Example 1

We wish to study the asymptotic behavior of $(\bar{x}, s)$ and to show that, $C(\bar{x}, s) \sim \beta_1\, \sigma^2/2k$, as obtained by Gumbel and Carlson (1956).

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By the Central Limit Theorem, if $\{X_i\}$ have moments up to the fourth order, we know that

$$A_k = \sqrt{k}\,\frac{\frac{1}{k}\sum_1^k x_i - \mu}{\sigma} \quad \text{and} \quad B_k = \sqrt{k}\,\frac{\frac{1}{k}\sum_1^k (x_i-\mu)^2 - \sigma^2}{\sqrt{\beta_2 - 1}\;\sigma^2}$$

are asymptotically standard normal and the random pair is asymptotically binormal with standard margins and correlation coefficient $\rho = \frac{\beta_1}{\sqrt{\beta_2-1}}$. But B_k, as $x_i - \mu = (x_i - \bar{x}) + (\bar{x} - \mu)$, takes the form

$$B_k = \sqrt{k}\,\frac{s^2 + (\bar{x}-\mu)^2 - \sigma^2}{\sqrt{\beta_2-1}\,\sigma^2}\,.$$

Then as $\sqrt{k}\,(\bar{x}-\mu)^2 = \sigma^2 A_k^2/\sqrt{k} \overset{P}{\rightarrow} 0$ and so

$$\left(\sqrt{k}\,\frac{\bar{x}-\mu}{\sigma}, \sqrt{k}\,\frac{s^2-\sigma^2}{\sqrt{\beta_2-1}\,\sigma^2}\right)$$

is also an asymptotically binomial pair with standard margins and correlation coefficient $\rho = \beta_1/\sqrt{\beta_2 - 1}$. Also $s^2 \overset{P}{\rightarrow} \sigma^2$ and, thus, $\frac{s^2-\sigma^2}{2\,\sigma(s-\sigma)} \overset{P}{\rightarrow} 1$.

Consequently $\left(\sqrt{k}\,\frac{\bar{x}-\mu}{\sigma}, 2\sqrt{k}\,\frac{s-\sigma}{\sqrt{\beta_2-1}\,\sigma}\right)$ is also an asymptotically binomial pair with standard margins, correlation coefficient ρ, and

$$C(\bar{x}, s) \sim \rho\,\frac{\sigma}{\sqrt{k}} \cdot \frac{\sqrt{\beta_2-1}\,\sigma}{2\sqrt{k}} = \frac{\beta_1\sigma^2}{2k}\,.$$

Example 2

Let $f(x, y)$ be a twice-differentiable function and $\{(X_k, Y_k)\}$ a sequence of random pairs such that $\left(\sqrt{k}\,\frac{X_k-\mu_x}{\sigma_x},\ \sqrt{k}\,\frac{Y_k-\mu_Y}{\sigma_y}\right)$ is asymptotically binormal with standard margins and correlation coefficient ρ. Then:

$\sqrt{k}\{f(X_k, Y_k) - f(\mu_x, \mu_y)\}$ *is asymptotically normal with mean value zero and variance* V *below.*

In fact, as

$$\sqrt{k}\{f(X_k, Y_k) - f(\mu_x, \mu_y)\} = \sqrt{k}\left\{(X_k - \mu_x)\frac{\partial f}{\partial x}\Big|_{\mu_x,\mu_y} + (Y_k - \mu_y)\frac{\partial f}{\partial y}\Big|_{\mu_x,\mu_y} + \text{2nd order term}\right\},$$

as the second order terms are $o_p(n^{-1/2})$ we see that $\sqrt{k}\{f(X_k, Y_k) - f(\mu_x, \mu_y)\}$ has the same asymptotic distribution of the linearized form

$$\sqrt{k}\{(X_k, \mu_x)\frac{\partial f}{\partial x}|_{\mu_x,\mu_y} + f(Y_k - \mu_y)\frac{\partial f}{\partial y}|_{\mu_x,\mu_y}\}$$

and is thus normal with mean value zero and variance

$$V = \sigma_x^2(\frac{\partial f}{\partial x}|_{\mu_x,\mu_y})^2 + 2\,\rho\,\sigma_x\,\sigma_y\frac{\partial f}{\partial z}|_{\mu_x,\mu_y}\,\frac{\partial f}{\partial y}|_{\mu_x,\mu_y} + \sigma_y^2(\frac{\partial f}{\partial y}|_{\mu_x,\mu_y})^2.$$

The multivariate generalization is obvious.

This result presupposes that not all the first derivatives are null; if this is not the case (i.e., all first derivatives are null) then we have different asymptotic distributions, such as the χ^2 — see Tiago de Oliveira (1982).

Reference

Cramér, H., 1946. *Mathematical Methods of Statistics*. Princeton Press, Princeton, NJ, 254-255.

Gumbel, E. J. and Carlson, P.G., 1956. On the asymptotic covariance between the sample mean and standard deviation, *Metron*, 18(1-2), 3-9.

Tiago de Oliveira, J., 1982. The δ-method for the obtention of asymptotic distributions — applications, *Inst. Statist. Univ. Paris*, 27, 49-70.

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