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Part 1
Probabilistic Patterns of Univariate Statistical Extremes

Exercises

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1.1 Show that $(\min\limits_{1}^{k} X_i) \xrightarrow{P} \underline{w}$.

1.2 Study the Cauchy distribution for the LLN.

1.3 Prove the conditions for the validity of MLLN.

1.4 Consider a sample of n observations $\{X_i\}$ and a possible future sample of m observations $\{Y_j\}$, all of them i.i.d. Show that

$$\text{Prob}\,\{\min\limits_{1}^{n} X_i \le \min\limits_{1}^{m} Y_j \le \max\limits_{1}^{m} Y_j \le \max\limits_{1}^{n} X_i\} = \frac{n(n-1)}{(n+m)(n+m-1)}\,.$$

1.5 Show that $\max\limits_{1}^{k}\{X_i\} \le a + \max\limits_{1}^{k}\{(X_i - a)_+$. If the $\{X_i\}$ have the margins $F_i(x)$ then $M(\max\limits_{1}^{k} X_i) \le a + \sum_1^k \int_a^{+\infty}(1 - F_i(x))d\,x$; no assumption of independence is made.

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1.6 Obtain the constants $(\alpha_m, \beta_m > 0)$ that satisfy the stability equation for maxima and for minima.

1.7 Obtain the transformations interchanging $\tilde{L}(x)$ between themselves, $\underset{\sim}{L}(x)$ between themselves and between $\tilde{L}(x)$ and $\underset{\sim}{L}(x)$; compare the different graphs of the densities to see the analogies.

1.8 Translate maxima results to minima results and vice-versa; use, in particular, the relation between the distribution function and the survival function and with the reduced Gumbel distribution.

1.9 $F(x) = 0$ if $x \le 0, F(x) = 1 - e^{-\sqrt{x}}$ is attracted for maxima to the Gumbel distribution; obtain one system of attraction coefficients.

1.10 Show that if $\overline{w} = +\infty$, and for some u and $v > 0$, as $x \to +\infty$, $x^u \exp(x^v)(1 - F(x)) \to c, 0 < c < +\infty$, then $\{(\max_1^k X_i - \lambda_k)/\delta_k\}$ is attracted to $\Lambda(x)$; a system of coefficients can be $\lambda_k = (\log(k\,c))^{1/v} - \frac{u \log \log (k\,c)}{V^2 (\log(k\,c))^{(v-1)/v}}$ and $\delta_k = \frac{(\log(k\,c))^{(1-v)/v}}{v}$.

Villasenor (1976)

1.11 Show that the conditions given for $F \in \mathcal{D}(\Lambda)$ are equivalent.

1.12 Obtain the continuous condition on $Q(v)$ for F to be attracted to the Gumbel distribution of minima.

1.13 The maximum likelihood estimators of (λ, δ) for the Gumbel distribution $\Lambda((x - \lambda)/\delta)$ are (Chapter 5, Part 2)

$$\hat{\delta} = \bar{x} - \frac{\sum_1^n x_i\, e^{-x_i/\hat{\delta}}}{\sum_1^n e^{-x_i/\hat{\delta}}}$$

$$\hat{\lambda} = -\,\hat{\delta} \log\left(\frac{1}{n} \sum_1^n e^{-x_i/\hat{\delta}}\right).$$

Show that if $\Phi(\theta) = \int_{-\infty}^{+\infty} e^{-\theta x}\, d\,\Lambda((x - \lambda)/\delta)$ then

$$\delta = -\Phi'(0) + \frac{\Phi'(1/\delta)}{\Phi(1/\delta)}$$

$$\lambda = -\delta \log\, \Phi(1/\delta)\,,$$

corresponding to the limiting structure of the maximum likelihood estimators.

1.14 If $F \in \mathcal{D}(\Lambda)$ then $F^k(x) \approx \Lambda(\frac{x-\lambda_k}{\delta_k})$. Define then $\Phi_k(\theta)$. Can this be used to obtain attraction coefficients? Can this be extended to the Weibull and Fréchet distributions?

1.15 Show that Fuller approximations to the return period of overpassing the level a for the Gumbel and Fréchet distributions for maxima are

$\widetilde{T}(a) = [-\log \Lambda(\frac{a-\lambda}{\delta})]^{-1} = e^{(a-\lambda)/\delta}$ and $\widetilde{T}(a) = [-\log \Phi_\alpha(\frac{a-\lambda}{\delta})]^{-1} = (\frac{a-\lambda}{\delta})^{\alpha}$ $(a > \lambda)$ for large a and that the return period of underpassing the level a for the Weibull distribution is $\widetilde{T}(a) = [-\log(1 - W_\alpha(\frac{a-\lambda}{\delta}))]^{-1} = (\frac{a-\lambda}{\delta})^{-\alpha}(a > \lambda)$ for small $a - \lambda$.

1.16 Show, for the Gumbel distribution, that $\widetilde{T}((1+\eta)a)$ and $\widetilde{T}(a)$ verify the relation $\widetilde{T}((1+\eta)a) = e^{\eta\,\lambda/\delta}\ \widetilde{T}^{1+\eta}(a)$. This approximation, with a very small relative error, shows the nonsense of the rule "to tame your river and build the dam strong enough to withstand anything, double the largest flood so far observed". Then $\eta = 1, \widetilde{T}(2\,a) = e^{\lambda/\delta}\ \widetilde{T}^2(a) > \widetilde{T}^2(a)$, as $\lambda > 0$, and so if the number of years of observation is 50 years, we have the return period $\widetilde{T}(2\,a) > 2500$ years which is not a realistic planning time.

Note: See in Part 2, for each limiting distribution, the status of statistical analysis of extreme data for risk evaluation and control and design planning.

1.17 Using the fact that, for the standard normal distribution, $\frac{1-N(x)}{N'(x)/x} \to 1$ as $x \to \infty$, and defining c_n by $c_n = n\,N'(c_n)$, show that $N^n(c_n + x/c_n) \to \Lambda(x)$.

1.18 Using Mill's inequalities for the standard normal distribution

$(\frac{1}{x} - \frac{1}{x^3})\,N'(x) < 1 - N(x) < \frac{1}{x}\,N'(x)$ for $x > 0$, show that $N^k(\lambda_k + \delta_k\, x) \to \Lambda(x)$ as $k \to \infty$ if $\lambda_k\, \delta_k \to 1$ and $\frac{k}{\sqrt{2\,\pi}\,\lambda_k}\, e^{-\lambda_k^2/2} \to 1$ as $k \to \infty$, thus directly obtaining the results in the text.

1.19 Let $\{X_k\}$ be a sequence of i.i.d. reduced Gumbel random variables. Fix an integer m and define $Y_k = \max(Y_i, X_{k+1}, \dots, X_{k+m-1})$. Show that $\text{Prob}\{Y_k - \log m \le x\} = \Lambda(x)$ and, also, that

$$\text{Prob}\,\{\max_1^k\,(Y_i - \log(k+m-1) \le x\} = \Lambda(x).$$

Note that the sequence $\{Y_k\}$ is $(m-1)$-dependent.

1.20 Let $\{X_i\}$ be a sequence of independent random variables such that $a_i\, X_i + b_i\ (a_i > 0)$ have the distribution function $F \in \mathcal{D}(\Lambda)$; let a system of attraction coefficients for F be (λ_k, δ_k). Show then, as

$F(y) = 1 - \frac{\exp(-(y-\lambda_k)/\delta_k)+0(1)}{k}$, that

$\text{Prob}\{(\max(X_1, \dots, X_k) - \lambda_k^{'})/\delta_k^{'} \le x\} \to \Lambda(x)$

If $\frac{1}{k}\sum_1^k e^{-(\eta_i + \eta_i\, x)} \to e^{-x} = 1$

where $\xi_i(= \xi_i(k)) = a_i\, \delta_k^{'}/\delta_k$ and $\eta_i(= \eta_i(k)) = (a_i\, \lambda_k^{'} + b_i - \lambda_k)/\delta_k$.

We can have the same attraction coefficients (i.e., $(\lambda_k^{'}, \delta_k^{'}) = (\lambda_k, \delta_k)$) if $\eta_i = 0$ and $\xi_i = 1$ or $a_i = 1$ and $b_i = 0$, which is the already classical case: but if $a_i = 1$ and $b_i = 0$ is false there may be a (non-equivalent) change in the attraction coefficients.

1.21 Compute the mean value and variance of Ψ_α and Φ_α (when they exist) up to $0(\alpha^{-1})$ and $0(\alpha^{-2})$ and obtain thus the linear transformation such that the transformed variables converge to the corresponding ones of the reduced Gumbel distribution (whose mean value and variance are γ and $\frac{\pi^2}{6}$) ; see, for more details, in Part 2, the Gumbel distribution.

1.22 Show that if Z_α has the reduced Weibull distribution for maxima, with shape parameter $\alpha > 0$, then $\alpha(1 + Z_\alpha)$ has the Gumbel distribution for maxima as a limit when $\alpha \to \infty$. Show also that if Z_α has the reduced

Fréchet distribution, with shape parameter $\alpha > 0$, then $\alpha(-1 + Z_\alpha)$ has the Gumbel distribution for maxima as a limit when $\alpha \to +\infty$.

1.23 The graphs suggest that $\Phi_\alpha(z)$ and $W_\alpha(z)$ are very close. Show, although, that

$$\sup_{z\geq 0}|W_\alpha(z) - \Phi_\alpha(z)| = |W_\alpha(1) - \Phi_\alpha(1)| = 1 - 2\,e^{-1} = .2642411,$$

With $W'_\alpha(1) = \Phi'_\alpha(1)$. Note that $\sup_{z\geq 0} = \left|W'_\alpha(z) - \Phi'_\alpha(z)\right| = +\infty$, for $\alpha < 1$ owing to the fact $W'_\alpha(0) = +\infty$, but for $\alpha \geq 1$ we have $\sup_{z\geq 0} = \left|W'_\alpha(z) - \Phi'_\alpha(z)\right| < +\infty$. Compute this bound.

1.24 As known, for the Fréchet distribution (for maxima) it can be shown that $\overline{w}_F = +\infty$ and that we can take $\lambda_k = 0$, i.e., $F^k(\delta_k\, z) \to \Phi_\alpha(z)$ as $k \to \infty$. Show that if $\text{Prob}\{\max(X_1, \dots, X_k)/\delta_k \leq z\} = F^n(\delta_n\, z) \to \Phi_\alpha(z)$, with $k \to \infty$, then with $Y_i = \log(\max(X_i, 0)) = \log(X_i)_+$ we have also $\text{Prob}\{\alpha(\max_1^k Y_i - \log \delta_k) \leq z\} \to \Lambda(z)$. Clearly the result stresses the importance of homogeneous estimators.

1.25 As known, for the Weibull distribution (for minima), it can be shown that $\underline{w}_F < +\infty$ and that we can take $\lambda_k = \underline{w}_F$, i.e., $1 - (1 - F(\underline{w}_F + \delta_k\, z))^k \to W_\alpha(z)$, as $k \to \infty$; Show that if $Y_i = -\log(X_i - \underline{w}_F)$ then $\text{Prob}\{\alpha(\max_1^k Y_i + \log \delta_k) \leq z\} \to \Lambda(z)$ also as $k \to \infty$. Consequently, in estimation, the left-end point is very important.

1.26 Using the relation between the Gumbel and Fréchet distributions, show that $F^n(\lambda_n + a\, z) \to \Lambda(z)$ if and only if $\frac{1-F(x)}{1-F(x+t)} \to e^{t/a}$ and so $1 - F(\log x)$ is a slowly varying function with index $1/a$.

Gnedenko (1943)

1.27 Show, in sequence to the previous exercise, that if X is normal (μ, σ) then $Y = X^2$ is also attracted to Gumbel distribution with $\delta_k = a = 2\,\sigma^2$.

1.28 For the examples 1 to 5 and for the attraction coefficients given in the section "The asymptotic distributions of extremes-some examples" (Chapter 2) compute numerically the probability and linear errors for probability levels $p = .95, .99, .999$ and values of $k = 10, 100, 1000$. Study the behaviour of example 6.

1.29 Use attraction conditions to show that both $F(x) = 0$ if $x < e$, $F(x) = 1 - \frac{1}{\log x}$ if $x \geq e$, and the Pascal distribution are not attracted to the Fréchet distribution.

1.30 Obtain the constants $(\alpha_t, \beta_t > 0)$ such that the stability equation for $G(z|\theta)$, i.e., $G^t(\alpha_t + \beta_t z|\theta) = G(z|\theta)$ is valid.

1.31 Show using the criteria that $G(z|\theta)$ is attracted to $G(z|\theta)$ as could be expected; i.e., Ψ_α to Ψ_α, Λ to Λ and Φ_α to Φ_α.

1.32 Show that the quantiles of $G(z|\theta)$ converge to those of the Gumbel distribution $\Lambda(z)$ when $\theta \to 0^+$ or $k \to 0^-$.

1.33 Study the case where the $\{X_i\}$ are i.i.d. and $\{X_i^*\}$, with $X_i^* = X_i + (-1)^i$, are both attracted for maxima (or for minima) to the same $\tilde{L}(x)$ ($\underset{\sim}{L}(x)$).

1.34 Show that if $\{X_i\}$ are i.i.d. such that $\overline{w} < +\infty$ and $\text{Prob}\{(\max_1^k X_i - \lambda_k)/\delta_k \leq x\} \to \tilde{L}(x)$, as $k \to \infty$, for $X_i^* = \frac{1}{\overline{w} - X_i}$ there exist λ_k^* and δ_k^* such that $\text{Prob}\{(\max_1^k X_i^* - \lambda_k^*)/\delta_k^* \leq x\}$ also converges to some $\tilde{L}(x)$.

1.35 Obtain conditions such that if $\{X_i\}$ are independent and such that if $\{(X_i - \lambda_i)/\delta_i\}$ have the same distribution $\tilde{L}(x)$, then $\max_1^k \{X_i\}$ also has the same asymptotic distribution $\tilde{L}(x)$; compute one system of attraction coefficients, asymptotically.

1.36 Using the fact that, if $F \in \mathcal{D}(\tilde{L})$ then $F(\lambda_k + \delta_k x) \to 1$ and if $F \in \mathcal{D}(\underset{\sim}{L})$ then $F(\lambda_k + \delta_k x) \to 0$, show, by the use of Khintchine's convergence of types theorem, that not only $(\lambda_{k+1}, \delta_{k+1})$ are also attraction coefficients for the same $\tilde{L}$ or $\underset{\sim}{L}$, i.e., $(\lambda_{k+1} - \lambda_k)/\delta_k \to 0$ and $\delta_{k+1}/\delta_k \to 1$ but, also, that $F^{n+k_n}(\lambda_n + \delta_n\, x) \to L(x)$ iff k_n/n; interpret the result.

1.37 Let $\{X_i\}$ be i.i.d. random variables with distribution function $F(x)$ attracted to $\tilde{L}(x)$ for maxima with attraction coefficients $\lambda_k, \delta_k > 0$ and to $\underset{\sim}{L}(x)$ for minima (with attraction coefficients $\lambda_k^{'}, \delta_k^{'} > 0$); recall that $\underset{\sim}{L}(x)$ may be different $1 - \tilde{L}(x)$, i.e., that the limiting distributions can not be in correspondence. It was shown that as $k \to \infty$

$$\text{Prob}\{\min_1^k X_i \le \lambda_k^{'} + \delta_k^{'}\, x, \max_1^k X_i \le \lambda_k + \delta_k\, y\}\ \underset{\sim}{L}(x)\ \tilde{L}(y).$$

Consider the range $R_k = X_k^{'} - X_1^{'}$ and show that, if $\delta_k^{'}/\delta_k \to b$,

$$\text{Prob}\{(R_k - (\lambda_k - \lambda_k^{'}))/\max(\delta_k, \delta_k^{'}) \le z\} \to \iint_D d\,\underset{\sim}{L}(x)\, d\,\tilde{L}(y)\,,$$

with $D = \{\frac{y}{\max(1,b)} - \frac{b\,x}{\max(1,b)} \le z\}$. Conclude the dominance of $X_k^{'}$ if $b = 0$, of $X_1^{'}$ if $b = +\infty$, and a joint influence (a convolution) if $0 < b < +\infty$; compare with the results of Chapter 8 (Analytical Statistical Choice for Univariate Extremes) of Part 2 for the Gumbel statistic Q.

Obtain specific results for $\Lambda(\frac{x-\lambda}{\delta})$, $\Phi_\alpha(\frac{x-\lambda}{\delta})$ and $W_\alpha(\frac{x-\lambda}{\delta})$ and other common distributions. Note that the δ-method can also be used.

Gumbel (1958), Galambos (1987)

1.38 Obtain corresponding results for the mid-point $(X_1^{'} + X_k^{'})/2$.

Gumbel (1958), Galambos (1987)

1.39 Obtain the corresponding results for the extremal quotient $EQ_k = X_k^{'}/X_1^{'}$; when the X_i are supposed to be non-negative ($\text{Prob}\{X_i > 0\} = 1$). Note that $\log EQ_k$ is the range of $\{\log X_i\}$.

Gumbel (1958)

1.40 Show that if $F \in \mathcal{D}(\tilde{L})$, with attraction coefficients (λ_k, δ_k) and $G(x)$ is such that $\frac{1-F(x)}{1-G(x)} \to c(0 < c < +\infty)$ when $x \to w_F$ then $G \in \mathcal{D}(\tilde{L})$ and a system of attraction coefficients is $(\lambda_{[ck]}, \delta_{[ck]})$.

1.41 Use the previous result to show that if X has a distribution function $F \in \mathcal{D}(\tilde{L})$ then the truncated random variable $X_a = \max(X, a)$, with $a < w_F$, has a distribution function $F_a \in \mathcal{D}(\tilde{L})$ also. Compute c.

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